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Orthotropic Swelling and Simplified Elasticity Laws with Special Reference to Cord-Reinforced Rubber

B. K. DANIELS, *Monsanto Company, Chemstrand Research Center, Inc., Durham, North Carolina 27702*

Synopsis

Further implications of equations for the directional dependence of swelling

$$a_\theta^2 = (a_T^2 - a_L^2)\sin^2\theta + a_L^2 \quad (1)$$

and elastic modulus

$$1/E_\theta = (\cos^2\theta/E_L) + (\sin^2\theta/E_T) \quad (2)$$

which were originated by Coran, Boustany, and Hamed[1] are given. Equation (1) is practically equivalent to the standard tensor transformation equation assuming that swelling is equivalent to negative hydrostatic pressure and the new relationship

$$\frac{a_L - 1}{a_T - 1} \approx \frac{E_T(1 - 2\nu_{LT})}{E_L(1 - \nu_{TT})}$$

is derived for the case $E_1 \gg E_2$. A corollary of eq. (1),

$$1/G_{LT} = 1/E_T + (1 + 2\nu_{LT})/E_L, \quad (3)$$

conflicts with three commonly used models of unidirectional composites. Anisotropic laminate theory is used to show that eq. (3) has important consequences for multiply laminates with no triangulation. These results indicate that the equations of Coran et al. cannot be expected to have wide application to other systems, especially continuous cord-reinforced rubber.

INTRODUCTION

The elastic constants and swelling properties of orthotropic short fiber–rubber composites have been studied by Coran, Boustany, and Hamed.[1] They showed experimentally that for their particular material, Young's modulus transformed with angle of measurement according to the unusually simple equation

$$1/E_\theta = \cos^2\theta/E_L + \sin^2\theta/E_T.$$

This implied theoretically, they pointed out, that the elastic constants referred to the principal axes were related by the equation

$$\frac{1}{G_{LT}} = \frac{1}{E_T} + \frac{(1 + 2\nu_{LT})}{E_L}. \quad (1)$$

Equation (1) is interesting in many respects, one of which is that it constitutes a reduction of the number of independent elastic constants from five to four. It is the purpose of the present paper to consider the implications of this equation if it should apply to orthotropic materials in general and to cord-reinforced rubber in particular. It is emphasized that no claims for any such generality were made by the above-mentioned authors.

Coran et al. derived a simple and direct theory of the relationship between elastic and swelling constants, eq. (2) below. It is shown here that the classic tensor treatment, with the assumption that swelling is equivalent to a negative hydrostatic pressure, gives the equivalent result (over the range of available data) and one other additional relationship. Equation (1) is contrasted with three commonly used models of unidirectional composites. It is also shown that eq. (1) holds certain implications for multidirectional laminates. These results are viewed as a development of the work of Coran et al. particularly to show the extent of applicability of their results to more complex systems.

DISCUSSION

Tensor Description of Swelling

The swelling relationship

$$a_\theta^2 = (a_T^2 - a_L^2)\sin^2\theta + a_L^2 \tag{2}$$

may be expressed in terms of infinitesimal strains, with which elasticity theory is concerned, as follows:

$$a_\theta^2 = a_L^2\cos^2\theta + a_T^2\sin^2\theta$$

where $a_\theta = 1 + \epsilon_\theta$; $a_L = 1 + \epsilon_L$; and $a_T = 1 + \epsilon_T$.
Since ϵ is small compared to unity,

$$\epsilon_\theta = \epsilon_L\cos^2\theta + \epsilon_T\sin^2\theta. \tag{3}$$

Equation (3) could have been obtained directly from the transformation of the strain tensor[2] given the characteristic swelling strains ϵ_L and ϵ_T. Equation (3) fits the data of Coran et al.[1] as well as eq. (2) (Fig. 1). Assuming swelling to be equivalent to a hydrostatic tension, σ, and since the compliance matrix for a transversely orthotropic material is

$$\begin{pmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{22} & 0 & 0 & 0 \\ (\text{Symm.}) & & & 2(S_{22} - S_{23}) & 0 & 0 \\ & & & & S_{66} & 0 \\ & & & & & S_{66} \end{pmatrix}$$

where $S_{11} = 1/E_L$, $S_{22} = 1/E_T$, $S_{12} = -\nu_{LT}/E_L$, and $S_{23} = -\nu_{TT}/E_T$, then

$$\epsilon_L = S_{11}\sigma + 2S_{12}\sigma$$

$$\epsilon_T = S_{22}\sigma + S_{23}\sigma + S_{12}\sigma.$$

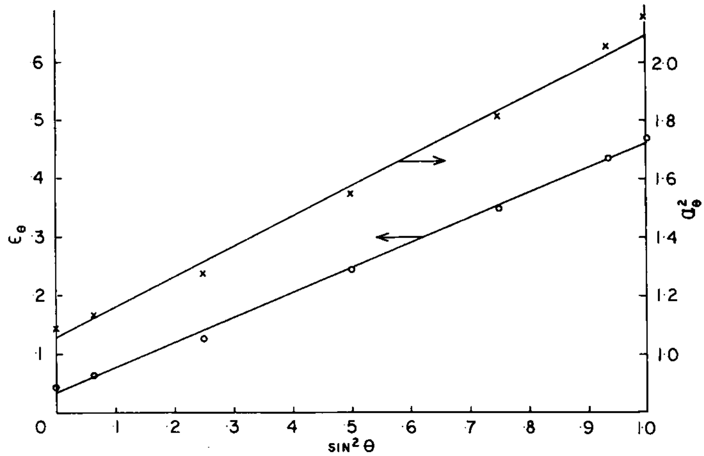

Fig. 1. Comparison of experimental swelling data with trigonometric (a_θ^2) and tensor (ϵ_θ) theories.

That is,

$$\epsilon_L = a_L - 1 = \sigma \frac{(1 - 2\nu_{LT})}{E_L}$$

and

$$\epsilon_T = a_T - 1 = \sigma \left(\frac{1 - \nu_{TT}}{E_T} - \frac{\nu_{LT}}{E_L}\right).$$

Thus if $E_L \gg E_T$,

$$\frac{\epsilon_L}{\epsilon_T} = \frac{E_T}{E_L} \frac{(1 - 2\nu_{LT})}{(1 - \nu_{TT})}.$$

This correspondence between the swelling constants and the elastic constants is subject to experimental verification.

Coran et al. derived eq. (1) from

$$E_\theta = \frac{K}{a_\theta - 1} \tag{4}$$

by substituting in eq. (2). This step is questionable because eq. (4) applies when the material is subjected to a uniaxial tensile stress alone, while eq. (2) applies when the material is subjected to a triaxial (hydrostatic) tension. Therefore, the general validity of eq. (1) is suspect. Alternatively, eq. (4) may be thought of as being correct only if K is a certain function of θ (as well as being proportional to the intensity of swelling).

Implications of Simplified Modulus-Angle Equation

Regardless of the way in which it was derived, the simplified expression

$$\frac{1}{E_\theta} = \frac{\cos^2\theta}{E_L} + \frac{\sin^2\theta}{E_T}$$

is substantiated[1] by three independent sets of experimental data, and eq. (1) follows from it via classical elasticity theory. It is of interest to examine the consequences of eq. (1) for more complex systems. The application to selected angle ply laminates of cord in rubber is described below. We may consider two specific ways in which eq. (1) can be satisfied by the elastic constants of a cord-rubber single ply. First, and most likely, ν_{LT} is supposed to be reasonably small (estimated values of cord rubber constants are shown in Table I) and $E_L \gg E_T$. Therefore,

$$G_{LT} \approx E_T. \tag{5}$$

Secondly, but this is physically unreasonable, we could use

$$\nu_{LT} = 2418 \tag{6}$$

together with the other values from Table I.

Equation (5) conflicts with a simple micromechanical model[3] in which the cords are taken to be rigid and square, of side C, and in which the rubber strain along the cord direction is zero. This model gives

$$E_T = \frac{(C + r)}{r}\,\frac{E_r}{1 - \nu_r^2} \tag{7}$$

where r is the width of rubber between adjacent cords, E_r is Young's modulus of the rubber, and ν_r is Poisson's ratio of the rubber. The model also gives

$$G_{LT} = \frac{C + r}{r}\,G_r. \tag{8}$$

Since

$$G_r = E_r/2(1 + \nu_r)$$

$$G_{LT} = E_T(1 - \nu_r)/2$$

and since

$$\nu_r = 0.5$$

$$G_{LT} = E_T/4 \tag{9}$$

TABLE I
Estimated Values of Cord-Rubber Ply Constants

E_L	$= 6.0\times10^6$ psi
E_T	$= 3.8\times10^3$ psi
G_{LT}	$= 9.6\times10^2$ psi
ν_{LT}	$= 0.38$

which, as stated above, conflicts with eq. (5). Any improvement in the micromechanics model would probably involve an extra restraint on the rubber strain in the direction perpendicular to the plane of the ply. This would increase the conflict between eqs. (1) and (5) still further.

It may also be noted that, according to the Halpin Tsai equations[4] for $G_c \gg G_r$, E_T/G_{LT} ranges from 3 to 4.5 as ν_c ranges from 0 to 1. For $G_c = G_r$, the range of E_T/G_{LT} is 3 to 8. (The cord is assumed to be transversely stiff.) Under no circumstances do the Halpin Tsai equations give $E_T = G_{LT}$.

Classic laminate theory[4] was applied to a selection of laminates under three conditions: (1) with the constants of Table I (where the square cord model has been used so that $E_T = 4G_{LT}$); (2) with the constants of Table I except $G_{LT} = E_T = 3.8 \times 10^3$ psi; and (3) with the constants of Table I

TABLE II
Theoretical Shear and Young's Moduli (10^3 psi) of a Variety of Laminates According to Three Different Relationships between Monoply Elastic Constants

Lay-up, degrees	(1) $E_T = 4G_{LT}$		(2) $E_T = G_{LT}$		(3) $\nu = 2418$	
	E	G	E	G	E	G
0	6000	0.96	6000	3.7	6000	0.96
5	124.5	0.98	462	3.7	511	0.96
10	32.4	1.05	124	3.7	137	0.96
20	15.0	1.38	32	3.7	36	0.95
40	3.2	3.48	9.0	3.7	9.7	0.94
60	2.8	2.14	5.0	3.7	5.2	0.94
90	3.7	0.96	3.8	3.7	3.7	0.96
±10	2255	176	2386	179	175	0.94
±20	184	620	262	622	45	0.91
±30	26.4	1125	60	1126	20	0.87
±45	3.8	1500	15	1500	9.2	0.83
90 ± 30	2002	751	2004	752	7.8	0.90
±20 ±30	741	873	778	874	28	0.89

except $\nu_{LT} = 2418$. The second and third of these conditions represent the two extreme ways, eqs. (5) and (6), of satisfying eq. (1) for cord in rubber. The choice of laminates and the results are shown in Table II. In each case, Young's modulus and the shear modulus of the laminate were calculated. This was done along various directions for the simple unidirectional laminates. (It was assumed that all cords are always in tension, and the tension-bending coupling term was neglected.)

For E, condition (2) tends to be equivalent to condition (1) when (a) the lay-up has some triangulation or (b) when the cords are along the reference direction. Conditions (a) and (b) are to be expected because they are the conditions for the stresses to be taken up largely by the cord rather than the rubber.

For G, conditions (1) and (2) differ only in the case of the monoply.

Even though condition (3) is physically unreasonable, it agrees with condition (2) for the E values of the monoplies. It gives wild values elsewhere. In particular, it does not come close to the approximate value[5]

$$\bar{E} = \frac{3}{8} E_L + \frac{5}{8} E_T = 2250$$

for the 90 ± 30 quasi-isotropic laminate.

For the monoplies, neither condition (2) nor condition (3) predicts a minimum E between 0° and 90°. This is contrary to physical expectation because it can be shown that with a rubber Poisson's ratio of 0.5, there would be no extension or compression of the cords if they were at 54.7° ($\tan^{-1}\sqrt{2}$) to the stress direction.

The net result to be drawn from these examples is that the proposed relationship, eq. (1), between elastic constants has important consequences for some laminated structures but not for others. For cord–rubber composites, eq. (1) seems to be contrary to two simple models of monoply behavior, contrary to the Halpin Tsai equations, and therefore its general validity is doubtful.

CONCLUSIONS

1. The trigonometric derivation of swelling at an arbitrary angle is equivalent, when strains are small, to the classical tensor treatment.

2. The longitudinal and transverse swelling constants should be related to each other by a function of the elastic constants:

$$\frac{\epsilon_L}{\epsilon_T} = \frac{a_L - 1}{a_T - 1} \approx \frac{E_T(1 - 2\nu_{LT})}{E_L(1 - \nu_{TT})} \text{ (when } E_1 \gg E_2).$$

3. The relationship

$$\frac{1}{G_{LT}} = \frac{1}{E_T} + \frac{(1 + 2\nu_{LT})}{E_L}$$

conflicts with the simple model of square rigid cords equally spaced in uniformly strained rubber.

4. The relationship (1) conflicts with the fact that cord rubber monoplies have minimum Young's modulus at $\tan^{-1}\sqrt{2}$.

5. The relationship (1) conflicts with the Halpin-Tsai equations.

6. Relationship (1) has important consequences for multiply laminates with no triangulation.

7. As originally proposed, relationship (1) should be applied only to certain unidirectional composites and, in view of 3, 4 and 5 above, its general validity for cord in rubber is doubtful.

Nomenclature

a extension ratio
C cord width
r rubber or width of rubber between cords
S_{ij} compliance matrix ($i,j = 1, 2, 3, 6$; l = longitudinal)
v volume fraction
E Young's modulus
$\bar{E}$ Young's modulus of quasi-isotropic laminate
G shear modulus
K constant
ϵ strain
ν Poisson's ratio (ν_{LT} refers to longitudinal stress)
θ angle between cord axis and direction of strain measurement
σ stress

Subscript

L longitudinal
T transverse
C cord

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